A New Approach to Optimal Impulsive Transfer Between Close Low-Eccentricity Elliptical Orbits

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Abstract

There can be situations in which it is desirable to transfer a spacecraft from a nearly circular orbit to another nearly circular orbit with minimum fuel expenditure. Some simple highly accurate formulas for such an impulsive transfer are presented in this paper. It is shown that two impulses are sufficient for optimality with this model. All of the non-degenerate optimal two-impulse solutions are found for this model and can be expressed in simple form. Examples of singular and nonsingular solutions are presented. There is a preponderance of singular solutions for this problem. The accuracy of the model is checked with two examples, one having an optimal singular solution, the other having an optimal nonsingular solution.

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1 Introduction

Impulsive orbital transfer between close low-eccentricity elliptical orbits that minimizes the total characteristic velocity is an important area of research with many potential applications. This classical problem has been addressed by distinguished early researchers including Lawden [1], Edelbaum [2], Kechichian [3], and Marec [4]. More recently the problem was discussed by Novoselov [5].

The present paper presents a highly accurate approximate closed-form solution of this problem that is simpler than previous solutions. The work follows a similar approach as that of optimal rendezvous near a nominal circular orbit [6] although the analysis is significantly different. The problem can be solved using a theory that was originally developed for linearized relative-motion problems [7]. We modify theorems from that reference to fit the present problem. Because of the assumption that the original orbits are near a circular orbit, we may replace increments of certain variables by differentials as in recent work [6] resulting in a final theorem yielding very simple formulas for singular and nonsingular solutions. The accuracy of the results can be justified mathematically, and is tested also through two examples with known solutions [8].

The primary results of this paper are to present a model whose solution requires only two impulses, is in reasonably simple closed form, and demonstrates a preponderance of singular solutions. Of secondary importance is the presentation of an approach that the authors think will be applicable to wider classes of optimal transfer and rendezvous problems.

2 A Review of the Model

We briefly outline the mathematical model presented in previous work[6].

2.1 The Impulse-Free Model

2.1.1 Changing the Independent Variable

The equation of motion of a spacecraft in an inverse-square gravitational field is

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the position vector of a spacecraft measured from a center of attraction, $r = |\mathbf{r}| = (\mathbf{r} \cdot \mathbf{r})^{1/2}$ is its magnitude, and μ is the product of the universal gravitational constant and a

point mass at the center of attraction, and the dot indicates differentiation with respect to time t. In polar coordinates (r, θ) this equation becomes

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{2}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}. \tag{3}$$

Multiplying (2) by r and integrating we obtain

$$r^2\dot{\theta} = h \tag{4}$$

where the constant h is the angular momentum of the spacecraft orbit. Using (4) to change the independent variable from t to θ and using a prime to denote differentiation with respect to θ the equations of motion can written as

$$r(\theta)r''(\theta) - 2r'(\theta)^2 = r(\theta)^2 - \mu r(\theta)^3/h^2 \tag{5}$$

$$h'(\theta) = 0. \tag{6}$$

These equations will be considered over a closed interval $\theta_0 \leq \theta \leq \theta_f$.

2.1.2 Linearizing the Equations

We shall employ a change of variable $y(\theta) = \frac{1}{r(\theta)}$ used in recent work. Based on this transformation we assign the new variables $y_1 = y$, $y_2 = -y'$, $y_3 = \frac{\mu}{h^2}$ and obtain the equivalent set of linear equations

$$y_1' = -y_2 \tag{7}$$

$$y_2' = y_1 - y_3 \tag{8}$$

$$y_3' = 0. \tag{9}$$

In recent work [6] we defined $y_1' = y_2$. We find that (7) used here is more convenient in geometric work [9]. The linearity of these equations is a useful feature. They can be written as

$$\mathbf{y}' = A\mathbf{y} \tag{10}$$

where $\mathbf{y} = (y_1, y_2, y_3)^T$ and

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

2.1.3 State Transition Matrix

A fundamental matrix solution

$$\Phi(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 1-\cos\theta \\ \sin\theta & \cos\theta & -\sin\theta \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

associated with (10) is seen to satisfy

$$\Phi'(\theta) = A\Phi(\theta), \quad \Phi(0) = I \tag{13}$$

where I is the 3×3 identity matrix. For arbitrary $\theta_1, \theta_2 \in [\theta_0, \theta_f]$ this defines a state-transition matrix $\Phi(\theta_2 - \theta_1)$ satisfying

$$\mathbf{y}(\theta_2) = \Phi(\theta_2 - \theta_1)\mathbf{y}(\theta_1), \tag{14}$$

consequently an impulse-free trajectory is propagated by

$$\mathbf{y}(\theta) = \Phi(\theta - \theta_0)\mathbf{y}(\theta_0) \tag{15}$$

for any for any real θ because (13) and (14) are independent of the end values θ_0 and θ_f. Letting $\mathbf{c}_0 = \Phi(-\theta_0)\mathbf{y}(\theta_0)$ and using properties of a fundamental matrix solution, an orbit through $\mathbf{y}(\theta_0)$ can be described as

$$\Phi(-\theta)\mathbf{y}(\theta) = \mathbf{c}_0. \tag{16}$$

At any value of θ the velocity of the spacecraft is

$$\mathbf{v} = (v_r, v_\theta)^T = (\dot{r}, r\dot{\theta})^T \tag{17}$$

In terms of the original variables the transformed variables are

$$y_1 = \frac{1}{r} \tag{18}$$

$$y_2 = \frac{\dot{r}}{h} \tag{19}$$

$$y_3 = \frac{\mu}{h^2} = \frac{\mu}{r^2 v_\theta^2}. \tag{20}$$

Values of the transformed variables at θ_0, and θ_f are obvious from those of the original variables at θ_0, and θ_f

2.2 The Impulsive Model

At $\theta_i \in [\theta_0, \theta_f]$ we consider an instantaneous jump in velocity (17) causing a jump in y_2 and y_3. Since r and θ are continuous there is no corresponding jump in y_1. Using the symbol Δ to indicate an instantaneous jump, the relationships between the jumps follow from (19) and (20)

$$\Delta y_{2i} = -\frac{1}{h_i}(\Delta v_{r_i} - \frac{v_{r_i}}{v_{\theta_i}}\Delta v_{\theta_i}), \;\; i = 1, ..., k \tag{21}$$

$$\Delta y_{3i} = -\frac{2\mu}{h_i^2}\frac{\Delta v_{\theta_i}}{v_{\theta_i}}, \;\; i = 1, ..., k. \tag{22}$$

In these expressions Δy_{2i} and Δy_{3i} are actually differentials obtained by differentiating (19)and (20). These representations are highly accurate for r_i, v_{ri}, and $v_{\theta i}$ near corresponding values on a nominal circular orbit. This is why these expressions could be used in [6]. In a later section we shall justify the use of (21) and (22) for initial and terminal orbits near a nominal circular orbit. We shall show that actual increments Δy_{2i} and Δy_{3i} respectively approach (21) and (22) as the boundary orbits become closer to a nominal circular orbit.

In the meantime, working with differences instead of differentials, in place of (21) and (22) we obtain

$$\Delta y_{2i} = -\frac{1}{h_i}\left(\frac{\Delta v_{ri} + \frac{v_{ri}}{v_{\theta_i}}\Delta v_{\theta_i}}{1 + \frac{\Delta v_{\theta_i}}{v_{\theta_i}}}\right), \;\; i = 1, \ldots, k \tag{23}$$

$$\Delta y_{3i} = -\frac{2\mu r_i \Delta v_{\theta_i}}{h_i^3}\frac{(1 + \frac{\Delta v_{\theta_i}}{2v_{\theta_i}})}{(1 + \frac{\Delta v_{\theta_i}}{v_{\theta_i}})^2}, \;\; i = 1, \ldots, k. \tag{24}$$

Employing these expressions, the initial and terminal orbits can be arbitrary; there is no need to restrict them to the vicinity of a nominal circular orbit.

Following [6] we introduce the transformed velocity increment

$$\Delta \mathbf{V}_i = (-\Delta y_{2i}, -\frac{1}{2}\Delta y_{3i})^T. \tag{25}$$

We now follow a development that appeared for impulsive linear systems in 1995 [7].

If there are no velocity jumps on an interval $(\theta_i, \theta]$ the state vector at θ is

$$\mathbf{y}(\theta) = \Phi(\theta - \theta_i)(\mathbf{y}(\theta_i) + B\Delta \mathbf{V}_i) \tag{26}$$

where

$$B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -2 \end{pmatrix}. \tag{27}$$

Similarly, if there are a total of k velocity jumps, the state vector at $\theta \in [\theta_k, \theta_f]$ is

$$\mathbf{y}(\theta) = \Phi(\theta - \theta_0)\mathbf{y}(\theta_0) + \Phi(\theta)\sum_{i=1}^{k} \Phi(-\theta_i)B\Delta\mathbf{V}_i. \tag{28}$$

Setting the boundary vector

$$\mathbf{z}_f = \Phi(-\theta_f)\mathbf{y}(\theta_f) - \Phi(-\theta_0)\mathbf{y}(\theta_0) \tag{29}$$

it follows that

$$\sum_{i=1}^{k} \Phi(-\theta_i)B\Delta\mathbf{V}_i = \mathbf{z}_f. \tag{30}$$

Having replaced the parameters Δv_i by ΔV_i, i=1,...,k we state the following problem. We seek to minimize the transformed total characteristic velocity

$$J = \sum_{i=1}^{k} |\Delta\mathbf{V}_i| \tag{31}$$

subject to (25)-(27), (29) and (30). To avoid trivial solutions we assume that $\mathbf{z}_f$ is not zero. We shall refer to this problem as the **transformed** problem.

3 Optimal Transformed Impulsive Rendezvous and Transfer Problems

In this section we define the primer vector and state some results on optimal transformed impulsive rendezvous and transfer problems.

3.1 The Primer Vector

As in previous work [6,7] we define the primer vector

$$\mathbf{p}(\theta) = R(\theta)^T\boldsymbol{\lambda} \tag{32}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^3$ and

$$R(\theta) = \Phi^{-1}(\theta)B. \tag{33}$$

For the magnitude of the primer vector we write $p(\theta) = |\mathbf{p}(\theta)| = (\mathbf{p}(\theta)^T\mathbf{p}(\theta))^{1/2}$. It follows from (12), (27), (32) and (33) that by writing $\mathbf{p}(\theta) = (p_1(\theta), p_2(\theta))^T$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^T$ we get

$$p_1(\theta) = \lambda_1 \sin\theta - \lambda_2 \cos\theta \tag{34}$$

$$p_2(\theta) = 2\lambda_1 \cos\theta + 2\lambda_2 \sin\theta - 2(\lambda_1 + \lambda_3). \tag{35}$$

Next we let $\lambda = (\lambda_1^2 + \lambda_2^2)^{1/2}$, $\lambda_1 = \lambda\cos\phi$, $\lambda_2 = \lambda\sin\phi$, $\lambda_3' = \lambda_1 + \lambda_3$ and obtain

$$p_1(\theta) = \lambda \sin(\theta - \phi) \tag{36}$$

$$p_2(\theta) = 2\lambda\cos(\theta - \phi) - 2\lambda_3'. \tag{37}$$

We note that $\mathbf{p}(\theta)$ is constant if and only if $\lambda = 0$. In all cases where $\mathbf{p}(\theta)$ is not constant we let $\kappa = \frac{\lambda_3'}{\lambda}$, $\bar{\theta} = \theta - \phi$. We arrive at the following form for the primer vector:

$$p_1(\theta) = \lambda \sin\bar{\theta} \tag{38}$$

$$p_2(\theta) = 2\lambda(\cos\bar{\theta} - \kappa) \tag{39}$$

and the magnitude is given by

$$p(\theta) = \lambda[\sin^2\bar{\theta} + 4(\cos\bar{\theta} - \kappa)^2]^{1/2}. \tag{40}$$

3.2 Optimal Fixed and Free Transformed Impulsive Transfer Problems

In a manner similar to previous work [6] we state the optimal k-impulse transformed rendezvous problem.

Given a positive integer k we seek a finite subset $K = \{\theta_1, \ldots, \theta_k\}$ of the set $[\theta_0, \theta_f]$ and velocity increments $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_k$ that minimize the transformed characteristic velocity (31) subject to the constraints (29) and (30) . We recall necessary conditions and sufficient conditions for normal optimal impulsive rendezvous [7]. For brevity we set $\alpha_i = |\Delta\mathbf{V}_i|$, $i = 1, \ldots, k$.

Theorem 1: If the set $K = \{\theta_1, \ldots, \theta_k\}$ and the velocity increments $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_k$ are minimizing solutions of the normal transformed impulsive rendezvous problem then it is necessary that

$$\Delta\mathbf{V}_i = 0, \;\; or \;\; \Delta\mathbf{V}_i = -\mathbf{p}(\theta_i)\alpha_i, \;\; i = 1, \ldots, k \tag{41}$$

$$\Delta\mathbf{V}_i = 0, \;\; or \;\; p(\theta_i) = 1, \;\; i = 1, \ldots, k \tag{42}$$

$$\theta_i = \theta_0, \;\; or \;\; p'(\theta_i) = 0, \;\; or \;\; \theta_i = \theta_f, \;\; i = 1, \ldots, k \tag{43}$$

$$\alpha_i \geq 0, \;\; i = 1, \ldots, k \tag{44}$$

$$\sum_{i=1}^{k} R(\theta_i)\mathbf{p}(\theta_i)\alpha_i = -\mathbf{z}_f. \tag{45}$$

These conditions strengthened by the condition

$$p(\theta) \leq 1, \;\; \theta_0 \leq \theta \leq \theta_f \tag{46}$$

are sufficient for the subset K and velocity increments $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_k$ to be a minimizing solution.

Proof: Sec. 3.2 and 3.3 of reference [7]. ■

Remark: Ref. [8] has some minor errors. In (24) of Sec. 3.2 it should read θ_i=θ_0 instead of θ_i=0. Also in the proof of Theorem 3.2 the negative sign should be removed in (41) and inserted in the second term of the expression that follows.

Remark: The sufficiency argument was presented in 1987 by Prussing and Clifton [10] and later repeated by Prussing [11]. It should be noted that (46) is not presented as a necessary condition here or in Sec. 3.3 of [7]. That proof is very similar to the proof of Theorem 2 that follows here.

Remark: The locus of the primer vector is on a unit circle if and only if $\lambda = 0$ and $\lambda_3' = \pm 1/2$ in (36) and (37). In this situation (42) and (43) are valid at arbitrary values of $\theta_1, \ldots, \theta_k$ on $[\theta_0, \theta_f]$ subject to (45), and equality holds in (46). A solution satisfying these properties is called **singular**. For solutions that are not singular, the primer vector takes the form (38) and (39).

Since an optimal impulsive transfer connects an initial orbit to a terminal orbit the first impulse occurs when $\mathbf{y}(\theta_1)$ is on an initial orbit (16), consequently

$$\Phi(-\theta_1)\mathbf{y}(\theta_1) = \mathbf{c}_I \tag{47}$$

where $\mathbf{c}_I$ defines the initial orbit. For any point $\mathbf{y}(\theta_I)$ on the initial orbit $\mathbf{c}_I = \Phi(-\theta_I)\mathbf{y}(\theta_I)$. Similarly

$$\Phi(-\theta_k)\mathbf{y}(\theta_k) = \mathbf{c}_F \tag{48}$$

where $\mathbf{c}_F = \Phi(-\theta_F)\mathbf{y}(\theta_F)$, and $\mathbf{y}(\theta_F)$ can be any point on the terminal orbit. The statement of the normal optimal transformed fixed k-impulse transfer problem is therefore the same as that of the normal optimal transformed k-impulse rendezvous problem except that (29) is replaced by

$$\mathbf{z}_f = \Phi(-\theta_k)\mathbf{y}(\theta_k) - \Phi(-\theta_1)\mathbf{y}(\theta_1) = \mathbf{c}_F - \mathbf{c}_I. \tag{49}$$

We use the terminology **fixed** if θ_0 and θ_f are specified apriori. For optimal transfer problems we are not usually interested in fixing θ_0 or θ_f. For the remainder of this paper we shall consider the normal optimal free transformed k-impulse transfer problem defined as follows.

Given a positive integer k, we seek a finite subset $K = \{\theta_1, \ldots, \theta_k\}$ of the real numbers and velocity increments $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_k$ that minimize the total transformed characteristic velocity subject to the constraints (30) and (47)-(49). The problem includes that of optimal transfer between elliptical orbits near a nominal circular orbit.

We present some necessary conditions and strengthen them to establish sufficient conditions for a solution of this problem.

Theorem 2: If the set $K = \{\theta_1, \ldots, \theta_k\}$ and the transformed velocity increments $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_k$ are minimizing solutions of the normal optimal transformed free k-impulse transfer problem, then it is necessary that

$$\Delta\mathbf{V}_i = 0, \;\; or \;\; \Delta\mathbf{V}_i = -\mathbf{p}(\theta_i)\alpha_i, \;\; i = 1, \ldots, k \tag{50}$$

$$p(\theta_i) = 1, \;\; and \;\; p'(\theta_i) = 0, \;\; i = 1, \ldots, k \tag{51}$$

$$\alpha_i \geq 0, \;\; i = 1, \ldots, k \tag{52}$$

$$\sum_{i=1}^{k} R(\theta_i)p(\theta_i)\alpha_i = -\mathbf{z_f} = -\mathbf{c_F} + \mathbf{c_I}. \tag{53}$$

These conditions strengthened by the condition

$$p(\theta) \leq 1, \;\; \theta_0 \leq \theta \leq \theta_f \tag{54}$$

are sufficient for $K = \{\theta_1, \ldots, \theta_k\}$ and $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_k$ to be a minimizing solution.

Proof: This proof is self-contained but it is a minor modification of that in [7]. Suppose $\theta_1^*, \ldots, \theta_k^*$ and the increments $\Delta\mathbf{V}_1^*, \ldots, \Delta\mathbf{V}_k^*$ are solutions of the minimization problem,then there exist multipliers $\lambda_0 \geq 0$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)^T$ not all zero so that the partial derivatives of the function

$$G = \lambda_0 \sum_{i=1}^{k} |\Delta\mathbf{V}_i| + \boldsymbol{\lambda}^T \left(\sum_{i=1}^{k} \Phi(-\theta_i) B \Delta\mathbf{V}_i - \mathbf{z}_f \right)$$

with respect to $\Delta V_{i1}, \Delta V_{i2}, \theta_i, \; i = 1, \ldots, k$ vanish at $\{\Delta\mathbf{V}_1^*, \ldots, \Delta\mathbf{V}_k^*, \theta_1^*, \ldots, \theta_k^*\}$. Since we are only considering normal problems we set $\lambda_0 = 1$. Using the terminology (32) and (33), the vanishing of the partial derivatives leads to (50) and (51); (52) follows from the definition of α_i, and satisfaction of the constraints (30) and (47)-(49) leads to (53).

For the sufficiency proof we let j be any positive integer, $\theta_1, \ldots, \theta_j$ arbitrary values on $\theta_0 \le \theta \le \theta_f$ and $\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_j$ arbitrary velocity increments associated with them, except that (30) and (47)-(49) are satisfied with j replacing k. We let $\theta_1^*, \ldots, \theta_k^*$, $\Delta\mathbf{V}_1^*, \ldots, \Delta\mathbf{V}_k^*$ satisfy (30) and (47)-(54). We shall show that

$$\sum_{i=1}^{k} |\Delta\mathbf{V}_i^*| \le \sum_{i=1}^{j} |\Delta\mathbf{V}_i|.$$

We take $\boldsymbol{\lambda}$ associated with $\Delta\mathbf{V}_1^*, \ldots, \Delta\mathbf{V}_k^*, \theta_1^*, \ldots, \theta_k^*$ and with it obtain $\mathbf{p}(\theta)$ from (32) so that (50)-(54) are satisfied for the starred set where $\theta_1^* \le \theta \le \theta_k^*$. We see from (38)-(40) that $p(\theta)$ is also defined on $\theta_1 \le \theta \le \theta_j$. In the singular case where $p(\theta) = 1$ identically this extension also applies. Using (30) associated with j and$\{\Delta\mathbf{V}_1, \ldots, \Delta\mathbf{V}_j, \theta_1, \ldots, \theta_j\}$ in view of (32) and (33), upon multiplying by $\boldsymbol{\lambda}$ we obtain

$$-\boldsymbol{\lambda}^T\mathbf{z}_f = -\sum_{i=1}^{j} \mathbf{p}(\theta_i)^T\Delta\mathbf{V}_i \le \sum_{i=1}^{j} p(\theta_i)|\Delta\mathbf{V}_i| \le \sum_{i=1}^{j} |\Delta\mathbf{V}_i|$$

where the last inequality follows from (54). The inequality (54) is also valid on the interval $\theta_1 \le \theta \le \theta_j$. Applying (30), (32) and (33) to $\Delta\mathbf{V}_1^*, \ldots, \Delta\mathbf{V}_k^*, \theta_1^*, \ldots, \theta_k^*$ yields

$$-\boldsymbol{\lambda}^T\mathbf{z}_f = \sum_{i=1}^{k} -\mathbf{p}(\theta_i^*)^T\Delta\mathbf{V}_i^* = \sum_{i=1}^{k} |\Delta\mathbf{V}_i^*|$$

where the last equality follows from (50). The result follows. ■

A set $K = \{\theta_1, \ldots, \theta_k\}$, increments $\Delta\mathbf{V}_1$, ..., $\Delta\mathbf{V}_k$, and a primer $\mathbf{p}(\theta)$ will be called **extremal** if they satisfy (50)-(53). An extremal is clearly optimal if it satisfies (54). Two extremals are called **equivalent** if they have the same cost (31). A k-impulse extremal that is equivalent to a j-impulse extremal is called **degenerate** if $j < k$. Some results in terms of $\bar{\mathbf{z}}_f = (\bar{z}_1, \bar{z}_2, \bar{z}_3)^T$ and $\mathbf{z}_f = (z_1, z_2, z_3)^T$ where $\bar{\mathbf{z}}_f = \Phi(\beta)\mathbf{z}_f$ are presented in the following theorem which is our main result.

Theorem 3: No more than two impulses are necessary to obtain nondegenerate optimal transfers. All optimal transfers are either:

(i) singular where

$$p_1(\theta) = 0, \quad p_2(\theta) = \pm 1 \tag{55}$$

and $\theta_1, \ldots, \theta_k$ are arbitrary on the interval $\theta_0 \le \theta \le \theta_f$ except that θ_1 satisfies (47), and θ_k satisfies (48), or

(ii) nonsingular where

$$p_1(\theta) = \pm\frac{1}{2}\sin\bar{\theta}, \quad p_2(\theta) = \pm\cos\bar{\theta} \tag{56}$$

and $\bar{\theta}_1$ and $\bar{\theta}_2$ are successive integer multiples of π or degenerate.

Optimal nonsingular solutions satisfy

$$\bar{z}_2 = 0, \tag{57}$$

$$\Delta\mathbf{V}_1 = \pm\alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Delta\mathbf{V}_2 = \pm\alpha_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{58}$$

and associated with the positive sign in (56) and (58) are

$$\alpha_1 = \frac{1}{4}(\bar{z}_1 - 2\bar{z}_3), \;\; \alpha_2 = \frac{1}{4}\bar{z}_1, \;\; J = \frac{1}{2}(\bar{z}_1 - 2\bar{z}_3); \tag{59}$$

associated with the negative sign are

$$\alpha_1 = -\frac{1}{4}(\bar{z}_1 - 2\bar{z}_3), \;\; \alpha_2 = -\frac{1}{4}\bar{z}_1, \;\; J = -\frac{1}{2}(\bar{z}_1 - 2\bar{z}_3). \tag{60}$$

All optimal two-impulse singular solutions satisfy

$$\begin{aligned} &\alpha_1\cos\bar{\theta}_1 + \alpha_2\cos\bar{\theta}_2 = \mp(\bar{z}_1 - \frac{1}{2}\bar{z}_3) \\ &\alpha_1\sin\bar{\theta}_1 + \alpha_2\sin\bar{\theta}_2 = \mp\frac{1}{2}\bar{z}_2 \\ &J = \alpha_1 + \alpha_2 = \pm\frac{1}{2}\bar{z}_3 \end{aligned} \tag{61}$$

where the upper and lower signs agree respectively with (55). The only optimal solutions for $\bar{z}_2 \neq 0$ are singular. An optimal two-impulse nonsingular solution cannot be equivalent to a singular solution. If $\bar{z}_2 = 0$ and $\bar{z}_3 = \frac{1}{2}\bar{z}_1$ or $\bar{z}_1 = 0$ optimal nonsingular solutions are of one impulse. Optimal nonsingular nondegenerate two-impulse solutions are confined to the region $\bar{z}_2 = 0$ and $\bar{z}_1 > 0$, $\bar{z}_3 > \frac{1}{2}\bar{z}_1$, or $\bar{z}_1 < 0$, $\bar{z}_3 < \frac{1}{2}\bar{z}_1$.

Proof: If $\lambda = 0$ in (36) and (37), then the necessary condition (51) requires that $\lambda_3' = \pm 1/2$ and (54) follows. Since $p(\theta) = \pm 1$ identically and can be extended over all of the interval $\theta_0 \leq \theta \leq \theta_f$ the necessary condition (51) is satisfied for the entire interval as long as the boundary conditions (30), (47)-(49) are satisfied, and solutions are singular.

If $\lambda \neq 0$ in (36) and (37), then these equations can be put in the form of (38) and (39), and the magnitude is given by (40). Differentiating this function and applying the necessary condition $p'(\theta_i) = 0$, $i = 1, \ldots, k$ from (51) we arrive at

$$\sin\bar{\theta}_i(-3\cos\bar{\theta}_i + 4\kappa) = 0.$$

It is necessary that the impulses occur where $\sin\bar{\theta} = 0$ or $\cos\bar{\theta} = \frac{4}{3}\kappa$. If an extremal has a non zero impulse at θ_i and also at $\theta_i + 2\pi$ then that extremal is degenerate by (53) because $R(\theta)$ and $p(\theta)$ have a period of 2π. Successive impulses can not occur where $\sin\bar{\theta} = 0$ and $\cos\bar{\theta}_{i+1} = \frac{4}{3}\kappa$ because the mean value theorem asserts that there exits $\hat{\theta}$ where $\bar{\theta}_i < \hat{\theta} < \bar{\theta}$ and for θ corresponding to $\hat{\theta}$ and $p'(\theta) = p(\theta_{i+1}) - p(\theta_i) = 0$ showing that either $\sin\hat{\theta} = 0$ or $\cos\hat{\theta} = \frac{4}{3}\kappa$ contradicting the assertion that θ_i and θ_{i+1} are successive. The same argument also shows that $\cos\bar{\theta}_i = \frac{4}{3}\kappa$ and $\sin\bar{\theta_{i+1}} = 0$ are forbidden. It follows from these arguments that successive impulses must satisfy either $\sin\bar{\theta}_i = \sin\bar{\theta}_{i+1} = 0$ or $\cos\bar{\theta}_i = \cos\bar{\theta}_{i+1} = \frac{4}{3}\kappa$. In the former case $p(\theta_i) = p(\theta_i + \pi) = 1$ implies $\kappa = 0$ and $\lambda = \pm 1/2$ resulting in (56). In the latter case $p(\theta_i) = 1$ implies that $\lambda = 1/(1 - \frac{4}{3}\kappa^2)^{1/2}$ resulting in

$$p_1(\theta) = \frac{\pm\sin^{\frac{1}{2}}\bar{\theta}}{1 - \frac{4}{3}\kappa^2}, \quad p_2(\theta) = \frac{\pm 2(\cos\bar{\theta} - \kappa)}{(1 - \frac{4}{3}\kappa^2)^{\frac{1}{2}}}.$$

For both this extremal and (56) the restriction of an impulse at both θ_i and also $\theta_i + 2\pi$ limits nondegenerate solutions to at most two impulses. It can be shown for this extremal that $p(\theta) \geq 1$, $\theta_0 \leq \theta \leq \theta_f$ but it cannot be excluded for this reason because we have not shown that (54) is necessary for optimality. That it is not optimal is shown below. We observe that $\bar{\mathbf{z}}_\mathbf{f} = \Phi(\beta)z_f$ for either extremal; solving this for β we get the same solution for either extremal.

We shall show that extremals associated with (56) are optimal. Since $\bar{\theta}_1$ and $\bar{\theta}_2$ are successive multiples of π there is no loss of generality by setting $\bar{\theta}_1 = 0$ and $\bar{\theta}_2 = \pi$. The condition (58) follows from (50) and the condition (53) becomes

$$R(0)\mathbf{p}(\theta_1)\alpha_1 + R(\pi)\mathbf{p}(\theta_2)\alpha_2 = -\bar{\mathbf{z}}_f.$$

Solving this set of three equations respecting the $\pm$ sign we obtain (57)-(60). Since these satisfy (50)-(53) and (56) satisfies (54) the result follows.

Next we show that the extremals characterized by $\cos\bar{\theta}_1 = \cos\bar{\theta}_2 = 4/3\kappa$ are not optimal. We consider the positive sign in (56) and (58). An analogous argument applies when we consider the negative sign.

Observing that $\bar{\theta}_1 = -\bar{\theta}_2$ we have $\sin\bar{\theta}_2 = -\sin\bar{\theta}_1 = (1-(4/3\kappa)^2)^{1/2}$ and

$$\mathbf{p}(\theta_1)^T = (-\lambda(1-(4/3\kappa)^2)^{1/2}, 2/3\lambda\kappa), \quad \mathbf{p}(\theta_2)^T = (\lambda(1-(4/3\kappa)^2)^{1/2}, 2/3\lambda\kappa).$$

As above condition (53) becomes

$$R(\bar{\theta}_1)\mathbf{p}(\theta_1)\alpha_1 + R(\bar{\theta}_2)\mathbf{p}(\theta_2)\alpha_2 = -\bar{\mathbf{z}}_f.$$

Solving this set of three equations we find again that $\bar{z}_2 = 0$ and

$$\lambda(\alpha_1 + \alpha_2) = \bar{z}_3 - \bar{z}_1.$$

Comparing this expression with the previous optimal result we find it necessary for optimality that $\lambda = 2$. Since $\lambda = 1/(1-4/3\kappa^2)^{1/2}$ this requires that $\kappa = \pm 3/2$ leading to the absurd result $\cos\bar{\theta}_1 = \cos\bar{\theta}_2 = \pm 2$.

Since $\bar{\mathbf{z}}_f = \Phi(\beta)\mathbf{z}_f$ also for singular solutions we may again use (53). Setting $k = 2$ this set of three equations produces (61). These equations allow for $\bar{z}_2 \neq 0$ although the nonsingular solutions do not.

If a singular solution is equivalent to a two-impulse nonsingular solution, then upon equating the cost, either $\bar{z}_3 - \bar{z}_1 = \bar{z}_3$ or $\bar{z}_3 - \bar{z}_1 = -\bar{z}_3$ so that either $\bar{z}_1 = 2\bar{z}_3$ or $\bar{z}_1 = 0$. In either case the nonsingular values (59), (60) lead to either $\alpha_1 = 0$ or $\alpha_2 = 0$, not a two-impulse solution.

The final statement follows form (59), (60) and the requirement that $\alpha_1 > 0$ and $\alpha_2 > 0$.

■

This theorem asserts that all nonsingular solutions are confined to the plane $\bar{z}_2 = 0$. For the positive sign in (56) and (58) nonsingular two-impulse solutions are found in the region $\bar{z}_1 > 0$, $\bar{z}_3 > \frac{1}{2}\bar{z}_1$. For the negative sign they are found in the region $\bar{z}_1 < 0$, $\bar{z}_3 < \frac{1}{2}\bar{z}_1$. Singular solutions are in the regions $\bar{z}_1 < 0$, $\bar{z}_3 > \frac{1}{2}\bar{z}_1$ or $\bar{z}_1 > 0$, $\bar{z}_3 < \frac{1}{2}\bar{z}_1$. The lines $\bar{z}_3 = \frac{1}{2}\bar{z}_1$, or $\bar{z}_1 = 0$ consist of one-impulse solutions except at the origin. This partitioning is depicted in Figure 1. The reader should note the preponderance of singular solutions.

4 Accuracy of the Approximation

The problem discussed in the preceding section was the minimization of the cost (31) subject to the constraints (29) and (30). This approximates the original problem of minimizing

$$J_0 = \sum_{i=1}^{k} |\Delta \mathbf{v}_i| \tag{62}$$

subject to these constraints where $\Delta \mathbf{v}_i$ is related to $\Delta \mathbf{V}_i$ through (23)-(25). Solution of this approximate problem with $k \leq 2$ satisfies the boundary conditions exactly and gives a highly accurate approximation of a minimum of (62) if the initial and terminal orbits are sufficiently close to a nominal circular orbit. Specifically, we shall choose a nominal circular orbit having orbital angular momentum

$$\bar{h} = \sqrt{\mu \bar{r}} \tag{63}$$

where $\bar{r}$ is the radius of this nominal circular orbit.

It follows from [6] that given an arbitrarily small real $\epsilon > 0$ there exists a real $\delta > 0$ such that minimizing values $\Delta \mathbf{v}_i^*$, $\Delta \mathbf{V}_i^*$, $i = 1, \ldots, k$ subject to (29) and (30) must satisfy

$$|\bar{h}|\Delta \mathbf{V}_i^*| - |\Delta \mathbf{v}_i^*|| \ < \frac{\epsilon}{k}, \quad i = 1, \ldots, k \tag{64}$$

if $|h_2 - h_1| < \delta$. As a consequence, the optimal costs J_0^* and J^* must satisfy

$$|\bar{h} J^* - J_0^*| < \epsilon. \tag{65}$$

To apply the work to an actual example we first set $k \leq 2$ according to Theorem 3, then determine whether a solution is singular or nonsingular and calculate nondegenerate solutions according to the appropriate formulas (55)-(61).

To illustrate we seek a solution in which J_0^* is known and compare it with $\bar{h} J^*$.

Example 1: The optimal solution is well known for the Hohmann Transfer in which the initial and terminal orbits are neighboring.

Since $\bar{\mathbf{z}}_f = \phi(\beta)\mathbf{z}_f$ and we can set $\beta = 0$ we have $\bar{\mathbf{z}}_f = \mathbf{z}_f$. Letting a_0 and a_f respectively denote the radii of the initial and terminal orbits we find $\mathbf{z}_f = \mathbf{c}_f - \mathbf{c}_0 = (\frac{1}{a_f} - \frac{1}{a_0})(1, 0, 1)^T$. It is evident from (59) and (60) that either α_1 or α_2 is negative, therefore this optimal solution must be singular! From the third equation of (61) we see that $J^* = \frac{1}{2}|\frac{1}{a_f} - \frac{1}{a_0}|$. Letting $\bar{a} = \frac{a_0 + a_f}{2}$ we get $\bar{h} = \sqrt{\mu \bar{a}}$. From the well-known Hohmann Transfer, we have $\Delta v_1^* = \sqrt{\frac{\mu}{a_0}}(\sqrt{\frac{a_f}{\bar{a}}} - 1)$ and $\Delta v_2^* = \sqrt{\frac{\mu}{a_f}}(1 - \sqrt{\frac{a_0}{\bar{a}}})$. Calculating $\bar{h} J^*$ and J_0^* in terms of $\sqrt{\frac{\mu}{a_0}}$ we find the error in replacing J_0^* by $\bar{h} J^*$ is approximately 0.14198% for $\frac{a_f}{a_0} = 1.1$. For $\frac{a_f}{a_0} = 1.4$ the error increases to approximately 1.77627%.

Example 2: Another example in which an optimal two-impulse solution is known consists of two identical elliptical orbits having a common central axis and opposing perigees as illustrated in Fig. 2. Optimal transfer consists of a circular orbit from apogee to apogee.[8]

In this example the initial and terminal orbits have apogee r_a, perigee r_p, orbital angular momentum h, and eccentricity e sufficiently small that the orbits can be considered nearly circular. The semi-major axis is $a = \frac{1}{2}(r_p + r_a)$ which is also chosen as the radius of the nominal circular orbit. The orbital angular momentum of the nominal circular orbit is therefore $\overline{h} = \sqrt{\mu a}$. The radius of the optimal two-impulse transfer orbit is $a_T = r_a$.

Setting $\beta = 0$ the initial orbit satisfies $y_1 = \frac{\mu}{h^2}(1 + e\cos(\theta - \pi))$, $y_2 = \frac{\mu e}{h^2}\sin(\theta - \pi)$, $y_3 = \frac{\mu}{h^2}$ whereas $y_1 = \frac{\mu}{h^2}(1 + e\cos\theta)$, $y_2 = \frac{\mu e}{h^2}\sin\theta$, $y_3 = \frac{\mu}{h^2}$ for the terminal orbit. We observe that $C_I = \Phi(-\pi)Y(\pi)$ and $C_F = \Phi(0)Y(0)$. Writing $Z_f = (z_1, z_2, z_3)^T$, we find that $z_1 = \frac{2\mu e}{h^2}$, $z_2 = z_3 = 0$. From (59) we observe that this solution is nonsingular and $J^\star = \frac{\mu e}{h^2}$. We note that $r_p = \frac{h^2/\mu}{1+e}$, $r_a = \frac{h^2/\mu}{1-e}$, and $a = \frac{h^2}{\mu(1-e^2)}$. It follows that $\bar{h} = \frac{h}{(1-e^2)^{\frac{1}{2}}}$ and $\bar{h}J^\star = \frac{\mu e}{h(1-e^2)^{\frac{1}{2}}}$. For the optimal transfer orbit $a_T = \frac{h^2}{\mu(1-e)}$ and its orbital angular momentum is $h^\star = \sqrt{\mu a_T} = \frac{h}{(1-e)^{\frac{1}{2}}}$. The optimal two-impulse cost [ ] is $J_0^\star = \frac{2\mu}{h^2}(h^\star - h)(1-e)$. This provides the formula $J_0^\star = \frac{2\mu}{h}[(1-e)^{\frac{1}{2}} - (1-e)]$.

We now compare $\bar{h}J^\star$ and $J_0^\star$ for $r_a/r_p = 1.1$. We get $e = 0.04761905$, $\frac{\bar{h}J^\star}{\mu/h} = 0.04767313$, $\frac{J_0^\star}{\mu/h} = 0.04703824$ and an error of 1.3497%. For the higher ratio $r_a/r_p = 1.4$ we have $e = 0.16666667$, $\frac{\bar{h}J^\star}{\mu/h} = 0.16903085$, $\frac{J_0^\star}{\mu/h} = 0.15907519$ with an error of 6.2535%.

Since the initial and terminal orbits intersect at $\theta = \pm\frac{\pi}{2}$ we consider a one-impulse solution. It is straight forward to show that the cost of a one-impulse solution is $J_1 = |\Delta V(\pm\frac{\pi}{2})| = \frac{2e\mu}{h}$. Clearly $J_0^\star < J_1$ for $e < 1$, and $\bar{h}J^\star < J_1$ for $e = \frac{\sqrt{3}}{2}$ which includes the near-circular orbits. A one-impulse solution therefore cannot be optimal for this class of problems. In fact $(J_1 - J_0)/J_0 = \frac{1}{(1-e)^{\frac{1}{2}}} > 1$ showing that a one-impulse solution has more than 100% error over an optimal two-impulse solution. The problem of fuel-optimal transfer between two elliptical orbits can be approximated and vastly simplified if the two orbits are near a nominal circular orbit and the approximation approaches exactness as the orbits approach this nominal circular orbit. Concise exact solutions are presented for this approximate model.

5 Conclusions

The problem of fuel-optimal transfer between two elliptical orbits can be approximated and vastly simplified if the two orbits are near a nominal circular orbit, and the approximation approaches exactness as the orbits approach this nominal circular orbit. Concise exact solutions are presented for this approximate model.

The accuracy of the model was calculated for two examples. The first, a Hohmann transfer with a ratio of 1.1 for the radii, was found to have an error of less than one seventh of one percent. The second transfer, between two identical ellipses with opposite perigees having an apogee-to-perigee ratio of 1.1 , was found to have an error of less than 1.5%. Since these orbits intersect, one-impulse solutions are possible but are not fuel efficient, having a cost more than twice that of the two-impulse solutions.

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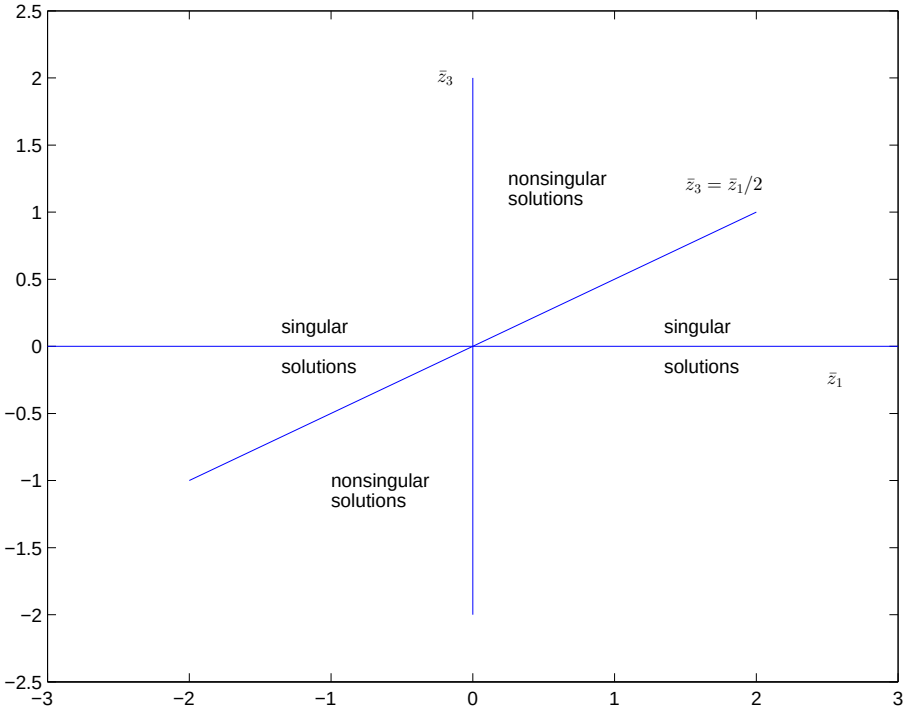

Figure 1: : Regions of singular and nonsingular solutions in the plane $\bar{z}_2 = 0$.

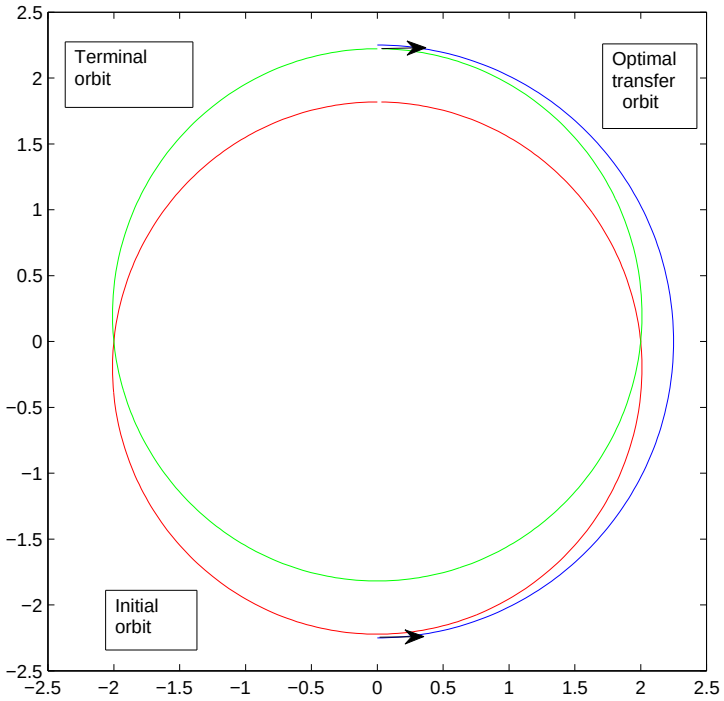

Figure 2: : Optimal transfer between identical near-circular orbits with opposing perigees.